

# HEIGHT OF SOME AUTOMORPHISMS OF LOCAL FIELDS

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ABSTRACT. In this note, we determine which automorphism subgroups of  $\text{Aut}_{\mathbb{F}_q}(\mathbb{F}_q((x)))$  are corresponding to  $\mathbb{Z}_p$ -extensions or  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extensions of characteristic 0 fields.

## 1. INTRODUCTION

Let  $k = \mathbb{F}_q$  be a finite field of characteristic  $p > 0$  and let  $L/K$  be a totally ramified abelian extension, where  $K$  is a local field with residue field  $k$ . Then  $G = \text{Gal}(L/K)$  has a decreasing filtration by the upper ramification subgroups  $G(r)$ , defined for nonnegative  $r \in \mathbb{R}$  (see [10, IV]). Since  $G$  is abelian,  $L/K$  is arithmetically profinite (see [12]). This means that for every  $r \geq 0$  the upper ramification group  $G(r)$  has finite index in  $G$ . This allows us to define the Hasse-Herbrand function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  where  $\psi_{L/F}(r) = \int_0^r [G : G(t)] dt$  and  $\phi_{L/K}(r) = \psi_{L/K}^{-1}(r)$ . The ramification subgroups of  $G$  with the lower numbering are defined by  $G[r] = G(\phi_{L/K}(r))$ .

Let  $\text{Aut}_k(k((x)))$  denote the group of continuous automorphisms of  $k((x))$  which induce the identity map on  $k$ . A closed abelian subgroup  $G$  of  $\text{Aut}_k(k((x)))$  also has a ramification filtration. The lower ramification subgroups of  $G$  are defined by

$$G[r] = \{\sigma \in G : v_x(\sigma(x) - x) \geq r + 1\}$$

for  $r \geq 0$ . Since  $G[r]$  has finite index in  $G$  for every  $r \geq 0$ , the function  $\phi_G : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  where  $\phi_G(r) = \int_0^r [G : G[t]]^{-1} dt$  is strictly increasing. We define the ramification subgroups of  $G$  with the upper numbering by  $G(r) = G[\phi_G^{-1}(r)]$ .

Wintenberger [11] has shown that the field of norms functor induces an equivalence between a category whose objects are totally ramified abelian  $p$ -adic Lie extensions  $L/K$ , where  $K$  is a local field with residue field  $k$ , and a category whose objects are pairs  $(\mathcal{K}, G)$ , where  $\mathcal{K} \simeq k((x))$  and  $G$  is an abelian  $p$ -adic Lie subgroup of  $\text{Aut}_k(\mathcal{K})$ . In short, if  $G$  is an abelian  $p$ -adic Lie subgroup of  $\text{Aut}_k(k((x)))$ , then there is an abelian  $p$ -adic Lie extensions  $L/K$  corresponding to  $(k((x)), G)$  by the equivalence of categories given by the field of norms functor. Moreover, the canonical isomorphism from  $\text{Gal}(L/K)$  onto  $G$  preserves the ramification filtration [6, 12]. This equivalence has been extended to allow  $\text{Gal}(L/K)$  and  $G$  to be arbitrary abelian pro- $p$  groups by Keating [3]. In the following, we will simply say that  $G$  is corresponding to  $L/K$  if the extension  $L/K$  corresponds to  $(k((x)), G)$  by the equivalence of categories given by the field of norms functor.

For  $\sigma \in \text{Aut}_k(k((x)))$ , we let

$$i(\sigma) = v_x \left( \frac{\sigma(x)}{x} - 1 \right).$$

Moreover, if  $\sigma(x) \equiv x \pmod{x^2}$ , then we denote  $i_n(\sigma) = i(\sigma^{p^n})$ . When  $\sigma \in \text{Aut}_k(k((x)))$  has infinite order, the sequence  $\{i_n(\sigma)\}$  is strictly increasing and attracts many attentions. In [9] Sen proved that for every  $n \in \mathbb{N}$ ,  $i_{n+1}(\sigma) \equiv i_n(\sigma) \pmod{p^{n+1}}$ . In [2] Keating determines upper bounds for the  $i_n(\sigma)$  in some cases and in [4, 5] the authors improve Keating's results using Wintenberger's theory of field of norms [11, 12]. These results are base on the fact that the automorphism subgroups correspond to  $\mathbb{Z}_p$ -extensions of characteristic 0 fields in [2, 4] and correspond to  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extensions of characteristic 0 fields in [5]. In this note, we determine which automorphism subgroups of  $\text{Aut}_k(k((x)))$  are corresponding to  $\mathbb{Z}_p$ -extensions or  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extensions of characteristic 0 fields. In the following, we will simply say that an extension  $L/K$  is of characteristic 0 if the characteristic of  $K$  is 0. Likewise, if the characteristic of  $K$  is  $p$ , then we say that the extension  $L/K$  is of characteristic  $p$ .

Motivated by the definition of height of a formal group and height of a  $p$ -adic dynamical system [7], we have the following definition.

**Definition 1.1.** Let  $\sigma \in \text{Aut}_k(k((t)))$  with  $\sigma \equiv x \pmod{x^2}$ . We say that the *height* of  $\sigma$  exists if  $\lim_{n \rightarrow \infty} i_n(\sigma)/i_{n-1}(\sigma)$  is finite and denote by

$$\text{Height}(\sigma) = \lim_{n \rightarrow \infty} \log_p \frac{i_n(\sigma)}{i_{n-1}(\sigma)}.$$

Let  $G$  be a closed subgroup of  $\text{Aut}_k(k((x)))$ . Our main result shows that if  $G$  is isomorphic to  $\mathbb{Z}_p$  then  $G$  corresponds to a characteristic 0 field extension if and only if every nonidentity element of  $G$  has height 1 and if  $G$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$  then  $G$  corresponds to a characteristic 0 field extension if and only if every nonidentity element of  $G$  has height 2.

The proof of our result is based on the following straightforward consequence of Theorem 4 of [8].

**Lemma 1.2.** Let  $L/K$  be an abelian extension and let  $G$  denote the Galois group  $\text{Gal}(L/K)$ .

- (1) If  $K$  is of characteristic  $p$ , then the mapping  $\sigma \rightarrow \sigma^p$  maps  $G(n)$  into  $G(pn)$ , for all  $n \in \mathbb{N}$ .
- (2) If  $K$  is of characteristic 0 with absolute ramification index  $e$ , then the mapping  $\sigma \rightarrow \sigma^p$  induces a homomorphism which maps  $G(n)/G(n+1)$  onto  $G(n+e)/G(n+e+1)$ , for all  $n$  large enough.

We remark that since  $G$  is abelian, every upper ramification break  $u$  (i.e.  $G(u) \supsetneq G(u+\epsilon)$ ,  $\forall \epsilon > 0$ ) is an integer (see for instance [10, V]). Therefore, we can apply Lemma 1.2 to the case where  $n$  is an upper ramification break of  $G$ . Moreover, if  $K$  is of characteristic 0 and  $\text{Gal}(L/K)$  is a pro- $p$  group, then Lemma 1.2 (2) shows that the mapping  $\sigma \rightarrow \sigma^p$  maps  $G(n)$  onto  $G(n+e)$ , for  $n$  sufficiently large. Therefore, in this case, if there is no nontrivial  $p$ -torsion element in  $G$ , then the mapping  $\sigma \rightarrow \sigma^p$  induces an isomorphism between  $G(n)/G(n+1)$  and  $G(n+e)/G(n+e+1)$ , for all  $n$  large enough. In particular, if  $n$  is large enough and  $n$  is an upper ramification break of  $G$ , then  $n+e$  is also an upper ramification break of  $G$ .

## 2. $\mathbb{Z}_p$ -EXTENSIONS

Given  $\sigma \in \text{Aut}_k(k((t)))$  with  $\sigma \equiv x \pmod{x^2}$ , write  $\lim_{n \rightarrow \infty} (i_n(\sigma)/p^n) = (p/(p-1))e$ . It is well-known that either  $e$  is a positive integer or  $e = \infty$  (see

for instance [13]). Moreover,  $e$  is a positive integer if and only if the field extension  $E/F$  corresponding to the closed subgroup generated by  $\sigma$  is of characteristic 0. In fact, in this case,  $e$  is the absolute ramification index of  $F$ . If  $e$  is finite, then it's clear that  $\lim_{n \rightarrow \infty} (i_n(\sigma)/i_{n-1}(\sigma)) = p$ . That is  $\text{Height}(\sigma) = 1$ . In this section, we will show that the converse is also true. Thus, for the case  $\sigma \in \text{Aut}_k(k((x)))$  with  $\text{Height}(\sigma) = 1$ , the closed cyclic group generated by  $\sigma$  corresponds to a  $\mathbb{Z}_p$ -extension of characteristic 0 field.

We prove this by contradiction. Suppose that the corresponding  $\mathbb{Z}_p$ -extension is of characteristic  $p$ . Then it is also true that the  $\mathbb{Z}_p$ -extension corresponding to the closed subgroup  $H$  generated by  $\sigma^{p^n}$  is of characteristic  $p$ . By considering the ramification groups of  $H$ , we have  $\sigma^{p^n} \in H[i_n(\sigma)] \setminus H[i_n(\sigma) + \epsilon]$  and  $\sigma^{p^{n+1}} \in H[i_{n+1}(\sigma)] \setminus H[i_{n+1}(\sigma) + \epsilon]$ ,  $\forall \epsilon > 0$ . Therefore  $\phi_H(i_n(\sigma))$  and  $\phi_H(i_{n+1}(\sigma))$  are upper ramification breaks of  $H$  and hence we can apply Lemma 1.2 (1) to get  $\sigma^{p^{n+1}} \in H(p \phi_H(i_n(\sigma)))$ . In other words,

$$\phi_H(i_{n+1}(\sigma)) = i_n(\sigma) + \frac{i_{n+1}(\sigma) - i_n(\sigma)}{p} \geq p \phi_H(i_n(\sigma)) = p i_n(\sigma), \forall n \in \mathbb{N}.$$

This says that

$$i_{n+1}(\sigma) \geq (p^2 - p + 1)i_n(\sigma), \forall n \in \mathbb{N},$$

and hence contradicts to the assumption that  $\lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{i_{n-1}(\sigma)} = p$ .

Conversely, suppose that  $G$  is corresponding to a characteristic 0 field extension  $E/F$  with  $e = v_F(p)$  being the absolute ramification index of  $F$ . By the definition of lower ramification group, for every  $n \in \mathbb{N}$ ,  $\sigma^{p^n} \in G[i_n(\sigma)] \setminus G[i_n(\sigma) + \epsilon]$  and  $\sigma^{p^{n+1}} \in G[i_{n+1}(\sigma)] \setminus G[i_{n+1}(\sigma) + \epsilon]$ , for every  $\epsilon > 0$ . On the other hand, by Lemma 1.2 (2) and the remark following it, when  $n$  is large enough if we let  $u = \phi_G(i_n(\sigma))$ , then  $u + e$  is an upper ramification break of  $G$ . Moreover, since  $\sigma^{p^n} \in G[i_n(\sigma)] = G(u)$ ,  $\sigma^{p^{n+1}} \in G(u)^p = \{g^p : g \in G(u)\} = G(u + e)$ . In other words,  $u + e = \phi_G(i_{n+1}(\sigma)) + e = \phi_G(i_{n+1}(\sigma))$ , and hence

$$e = \phi_G(i_{n+1}(\sigma)) - \phi_G(i_n(\sigma)) = \frac{1}{p^{n+1}}(i_{n+1}(\sigma) - i_n(\sigma))$$

because  $G$  is isomorphic to  $\mathbb{Z}_p$ . This is true for all  $n$  large enough. Therefore, we conclude that there exists  $m \in \mathbb{N}$  such that for all  $n > m$ ,

$$\begin{aligned} i_n(\sigma) &= i_m(\sigma) + \sum_{j=m+1}^n (i_j(\sigma) - i_{j-1}(\sigma)) \\ &= i_m(\sigma) + e(p^{m+1} + \cdots + p^n) \\ &= i_m(\sigma) + \frac{ep}{p-1}(p^n - p^m). \end{aligned}$$

This shows

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{p^n} = \frac{ep}{p-1}.$$

We summarize this result as the following.

**Theorem 2.1.** *Suppose that  $G \subseteq \text{Aut}_k(k((x)))$  is a closed subgroup generated by  $\sigma$  which is isomorphic to  $\mathbb{Z}_p$ . Then the following are equivalent:*

$$(1) \lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{i_{n-1}(\sigma)} = p$$

- (2) The sequence  $\{\frac{i_n(\sigma)}{p^n}\}_n$  converges.
- (3)  $\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p$  for all  $n$  sufficiently large.
- (4) The  $\mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0.

*Remark 2.2.* Theorem 2.1 remains true if we replace  $\sigma$  in the statements (1), (2) and (3) by any nonidentity element  $\tau \in G$ . This is because the closed subgroup of  $G$  generated by  $\tau$  is a finite index subgroup. In other words, we shows that the  $\mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0 if and only if every nonidentity element  $\tau \in G$  has  $\text{Height}(\tau) = 1$ .

### 3. $\mathbb{Z}_p \times \mathbb{Z}_p$ -EXTENSIONS

In this section we extend the result of the previous section to the case that  $G \subseteq \text{Aut}_k(k((x)))$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$ . In this case we show that every nonidentity element of  $G$  has height 2 if and only if the  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0.

Let  $G$  be a closed subgroup of  $\text{Aut}_k(k((x)))$  which is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$  and suppose that for every nonidentity element  $\sigma \in G$  we have  $\lim_{n \rightarrow \infty} i_n(\sigma)/i_{n-1}(\sigma) = p^2$ . Again, we use method of contradiction to show that the  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0. First, suppose that the corresponding  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension is of characteristic  $p$ . Then for any two linearly independent elements  $\sigma, \tau \in G$ , since  $\langle \sigma, \tau \rangle$  is a finite index subgroup of  $G$  (we use  $\langle \sigma, \tau \rangle$  to denote the closed subgroup of  $G$  generated by  $\sigma$  and  $\tau$ ), the field extension corresponding to  $\langle \sigma, \tau \rangle$  is also a characteristic  $p$  field extension. Similarly, for  $m, n \in \mathbb{N}$ , the  $\mathbb{Z}_p \times \mathbb{Z}_p$  extension corresponding to  $\langle \sigma^{p^n}, \tau^{p^m} \rangle$  is also of characteristic  $p$ . We consider several cases.

For a given nonidentity  $\sigma \in G$ , we first suppose that for every  $N \in \mathbb{N}$ , there exist  $n, m > N$  and  $\tau$  of  $G$  such that  $i_n(\sigma) < i_m(\tau) < i_{m+1}(\tau) \leq i_{n+1}(\sigma)$ . Notice that the field extension corresponding to the closed subgroup  $H = \langle \sigma^{p^n}, \tau^{p^m} \rangle$  is also of characteristic  $p$ . By considering the lower ramification subgroups of  $H$ , we have

$$\begin{aligned} H[i_n(\sigma)] = \langle \sigma^{p^n}, \tau^{p^m} \rangle \supsetneq H[i_n(\sigma) + 1] = \cdots = H[i_m(\tau)] = \langle \sigma^{p^{n+1}}, \tau^{p^m} \rangle \\ \supsetneq H[i_m(\tau) + 1] = \cdots = H[i_{m+1}(\tau)] = \langle \sigma^{p^{n+1}}, \tau^{p^{m+1}} \rangle. \end{aligned}$$

Therefore,

$$\phi_H(i_m(\tau)) = i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p}$$

and

$$\phi_H(i_{m+1}(\tau)) = i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p} + \frac{i_{m+1}(\tau) - i_m(\tau)}{p^2}.$$

Since  $\tau^{p^{m+1}} \in H[i_m(\tau)] = H(\phi_H(i_m(\tau)))$  and

$$\tau^{p^{m+1}} \in H(\phi_H(i_{m+1}(\tau))) \setminus H(\phi_H(i_{m+1}(\tau)) + \epsilon), \quad \forall \epsilon > 0,$$

Lemma 1.2 (1) says

$$i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p} + \frac{i_{m+1}(\tau) - i_m(\tau)}{p^2} \geq p(i_n(\sigma) + \frac{i_m(\tau) - i_n(\sigma)}{p}).$$

Therefore by  $i_{n+1}(\sigma) \geq i_{m+1}(\tau)$  and  $i_m(\tau) > i_n(\sigma)$ , we have

$$i_{n+1}(\sigma) \geq (p^2 - p + 1)i_m(\tau) + (p^3 - 2p^2 + p)i_n(\sigma) > (p^3 - p^2 + 1)i_n(\sigma).$$

Since for every  $N \in \mathbb{N}$ , this is true for some  $n > N$ , it contradicts to the assumption that  $\lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{i_{n-1}(\sigma)} = p^2$ .

Now suppose that for every  $N \in \mathbb{N}$ , there exist  $n, m > N$  and  $\tau$  of  $G$  such that  $i_n(\sigma) < i_m(\tau) < i_{n+1}(\sigma) < i_{n+2}(\sigma) \leq i_{m+1}(\tau)$ . Notice that the field extension corresponding to the closed subgroup  $H = \langle \sigma^{p^n}, \tau^{p^m} \rangle$  is also of characteristic  $p$ . By considering the lower ramification subgroups of  $H$ , we have

$$\begin{aligned} H[i_n(\sigma)] = \langle \sigma^{p^n}, \tau^{p^m} \rangle &\supsetneq H[i_n(\sigma) + 1] = \cdots = H[i_m(\tau)] = \langle \sigma^{p^{n+1}}, \tau^{p^m} \rangle \\ &\supsetneq H[i_m(\tau) + 1] = \cdots = H[i_{n+1}(\sigma)] = \langle \sigma^{p^{n+1}}, \tau^{p^{m+1}} \rangle \\ &\supsetneq H[i_{n+1}(\sigma) + 1] = \cdots = H[i_{n+2}(\sigma)] = \langle \sigma^{p^{n+2}}, \tau^{p^{m+1}} \rangle. \end{aligned}$$

Therefore

$$\phi_H(i_{n+2}(\sigma)) = \phi_H(i_{n+1}(\sigma)) + \frac{i_{n+2}(\sigma) - i_{n+1}(\sigma)}{p^3}.$$

By Lemma 1.2 (1),  $\phi_H(i_{n+1}(\sigma)) \geq p\phi_H(i_n(\sigma)) = pi_n(\sigma)$  and since  $\sigma^{p^{n+2}} \notin H[i_{n+2}(\sigma) + \epsilon]$ ,  $\forall \epsilon > 0$ , we have  $\phi_H(i_{n+2}(\sigma)) \geq p\phi_H(i_{n+1}(\sigma))$ . This implies

$$\frac{i_{n+2}(\sigma) - i_{n+1}(\sigma)}{p^3} \geq (p-1)\phi_H(i_{n+1}(\sigma)) \geq (p-1)pi_n(\sigma),$$

and hence

$$i_{n+2}(\sigma) \geq (p^5 - p^4)i_n(\sigma) + i_{n+1}(\sigma).$$

Since for every  $N \in \mathbb{N}$ , this is true for some  $n > N$ , it contradicts to the assumption that  $\lim_{n \rightarrow \infty} \frac{i_{n+1}(\sigma)}{i_n(\sigma)} = p^2$ .

Now we only have the following two cases to consider:

- (1) There exists  $N$  such that there is neither  $m, n > N$  nor any nonidentity  $\tau \in G$  such that  $i_n(\sigma) < i_m(\tau) < i_{n+1}(\sigma)$ .
- (2) There exists  $m, n \in \mathbb{N}$  and a nonidentity  $\tau \in G$  such that  $i_{n+j}(\sigma) < i_{m+j}(\tau) < i_{n+j+1}(\sigma)$ , for all  $j \in \mathbb{N}$ .

For the case (1), there exists  $N \in \mathbb{N}$  such that

$$G[i_n(\sigma)] \supsetneq G[i_n(\sigma) + 1] = \cdots = G[i_{n+1}(\sigma)], \forall n > N.$$

Now let  $H = G[i_n(\sigma)]$ . Then by the contrapositive assumption, the field extension corresponding to  $H$  is also of characteristic  $p$ . Since  $\phi_H(i_{n+1}(\sigma)) = i_n(\sigma) + (1/p^2)(i_{n+1}(\sigma) - i_n(\sigma))$ , again by Lemma 1.2 (1) we have  $i_n(\sigma) + (1/p^2)(i_{n+1}(\sigma) - i_n(\sigma)) \geq pi_n(\sigma)$  and hence

$$i_{n+1}(\sigma) \geq (p^3 - p^2 + 1)i_n(\sigma) \geq (p^2 + 1)i_n(\sigma).$$

This is true for all  $n > N$ , and hence it contradicts to the assumption that  $\text{Height}(\sigma) = 2$ .

For the case (2), for every  $j \in \mathbb{N}$ , let  $H = \langle \sigma^{p^{n+j}}, \tau^{p^{m+j}} \rangle$  and by considering the ramification subgroups of  $H$ , we have

$$\phi_H(i_{n+j+1}(\sigma)) = i_{n+j}(\sigma) + \frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{p} + \frac{i_{n+j+1}(\sigma) - i_{m+j}(\tau)}{p^2}.$$

Again, by the contrapositive assumption, the field extension corresponding to  $H$  is of characteristic  $p$ , and hence by Lemma 1.2 (1)

$$\phi_H(i_{n+j+1}(\sigma)) \geq p\phi_H(i_{n+j}(\sigma)) = pi_{n+j}(\sigma).$$

This says that

$$(3.1) \quad i_{n+j+1}(\sigma) \geq (p^3 - p^2 + p)i_{n+j}(\sigma) - (p-1)i_{m+j}(\tau).$$

Since  $\lim_{j \rightarrow \infty} i_{n+j+1}(\sigma)/i_{n+j}(\sigma) = p^2$ , for every  $1 > \epsilon > 0$ , there exists  $j$  large enough such that  $p^2 - \epsilon < i_{n+j+1}(\sigma)/i_{n+j}(\sigma) < p^2 + \epsilon$ . Similarly,  $p^2 - \epsilon < i_{m+j+1}(\tau)/i_{m+j}(\tau) < p^2 + \epsilon$ . Hence, we can have either  $i_{m+j}(\tau) < (p+\epsilon)i_{n+j}(\sigma)$  or  $i_{n+j+1}(\sigma) < (p+\epsilon)i_{m+j}(\tau)$ . Otherwise  $i_{m+j}(\tau) \geq (p+\epsilon)i_{n+j}(\sigma)$  and  $i_{n+j+1}(\sigma) \geq (p+\epsilon)i_{m+j}(\tau)$  imply  $i_{n+j+1}(\sigma) \geq (p+\epsilon)^2 i_{n+j}(\sigma) > (p^2 + \epsilon)i_{n+j}(\sigma)$ . Without loss of generality (switching  $\sigma$  and  $\tau$  if necessary), for every  $N \in \mathbb{N}$  and  $1 > \epsilon > 0$ , we can find  $j > N$  such that  $i_{m+j}(\tau) < (p+\epsilon)i_{n+j}(\sigma)$  and hence by Equation (3.1), we get

$$i_{n+j+1}(\sigma) \geq (p^3 - p^2 + p)i_{n+j}(\sigma) - (p-1)(p+\epsilon)i_{n+j}(\sigma).$$

Thus

$$i_{n+j+1}(\sigma) \geq (p^3 - 2p^2 + (2-\epsilon)p + \epsilon)i_{n+j}(\sigma).$$

This contradicts to the assumption that  $\lim_{j \rightarrow \infty} \frac{i_{n+j+1}(\sigma)}{i_{n+j}(\sigma)} = p^2$ , for  $p \geq 3$ .

For the case  $p = 2$ , considering the ramification subgroups

$$H[i_{n+j}(\sigma)] \supsetneq H[i_{m+j}(\tau)] \supsetneq H[i_{n+j+1}(\sigma)] \supsetneq H[i_{m+j+1}(\tau)] \supsetneq H[i_{n+j+2}(\sigma)],$$

we have

$$\begin{aligned} \phi_H(i_{n+j+2}(\sigma)) &= i_{n+j}(\sigma) + \frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{2} + \frac{i_{n+j+1}(\sigma) - i_{m+j}(\tau)}{4} \\ &\quad + \frac{i_{m+j+1}(\tau) - i_{n+j+1}(\sigma)}{8} + \frac{i_{n+j+2}(\sigma) - i_{m+j+1}(\tau)}{16}. \end{aligned}$$

Again by the assumption that the corresponding field extension is of characteristic 2, we have  $\phi_H(i_{n+j+2}(\sigma)) \geq 2\phi_H(i_{n+j+1}(\sigma))$  and deduce that

$$i_{n+j+2}(\sigma) \geq 8i_{n+j}(\sigma) + 4i_{m+j}(\tau) + 6i_{n+j+1}(\sigma) - i_{m+j+1}(\tau).$$

Again, without loss of generality, for every  $N \in \mathbb{N}$  and  $1 > \epsilon > 0$ , we can assume there exists  $j > N$  such that  $i_{n+j+1}(\sigma) > (4-\epsilon)i_{n+j}(\sigma)$ ,  $i_{m+j+1}(\tau) < (4+\epsilon)i_{m+j}(\tau)$  and  $i_{m+j}(\tau) < (2+\epsilon)i_{n+j}(\sigma)$ . Therefore, by using  $i_{m+j}(\tau) > i_{n+j}(\sigma)$ , we get

$$i_{n+j+2}(\sigma) > (28 - 12\epsilon - \epsilon^2)i_{n+j}(\sigma).$$

This contradicts to the assumption that  $\lim_{j \rightarrow \infty} \frac{i_{n+j+2}(\sigma)}{i_{n+j}(\sigma)} = 2^4$ . We complete the proof of showing that if every nonidentity element of  $G$  is of height 2, then the  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0.

Conversely, suppose the field extension  $E/F$  corresponding to  $G$  is of characteristic 0 with  $e = v_F(p)$  being the absolute ramification index of  $F$ . Then since there is no  $p$ -torsion element in  $G$ , by Lemma 1.2 (2) and the remark following it, there exists an  $N$  such that the raise to  $p$ -th power map  $G(u)/G(u+\epsilon) \mapsto G(u+e)/G(u+e+\epsilon)$  is an isomorphism for all  $u > N$ . In other words, there exists  $N \in \mathbb{N}$  such that for every upper ramification break  $u > N$ ,  $[G(u) : G(u+\epsilon)]$  is either always 2 or always 1. For simplicity, we call the former *depth 2* case and the latter *depth 1* case.

For depth 2 case, it means that for every  $\sigma, \tau \in G$ , there exists  $n, m \in \mathbb{N}$  such that  $i_{n+j}(\sigma) = i_{m+j}(\tau)$  for all  $j \in \mathbb{N}$ . Therefore, for every  $\sigma \in G$ , we choose another  $\tau \in G$  so that  $G[i_n(\sigma)] = G(u_1) = \langle \sigma^{p^n}, \tau^{p^m} \rangle$ ,  $G[i_{n+1}(\sigma)] = G(u_2) = \langle \sigma^{p^{n+1}}, \tau^{p^{m+1}} \rangle$ , where  $u_1, u_2$  are upper ramification breaks and  $G(u)^p = G(u+e)$  for all  $u \geq u_1$ . Let  $G'$  be the closed subgroup of  $G$  generated by  $\sigma$  and  $\tau$ . It is clear that  $G'$  is of finite index over  $G$  and hence the field extension  $E/F'$  corresponding to  $G'$

is also of characteristic 0. Let  $e'$  be the absolute ramification index of  $F'$ . Since  $G'[i] = G[i] \cap G'$ , we get  $\phi_{G'}(i_{n+1}(\sigma)) - \phi_{G'}(i_n(\sigma)) = (i_{n+1}(\sigma) - i_n(\sigma))/p^{n+m+2} = e'$ . Inductively, we have

$$(3.2) \quad \frac{i_{n+j}(\sigma) - i_{n+j-1}(\sigma)}{p^{n+m+2j}} = e'.$$

This shows

$$\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p^2$$

for all  $n$  large enough. Moreover, since

$$i_{n+j}(\sigma) = i_n(\sigma) + (i_{n+1}(\sigma) - i_n(\sigma)) + \cdots + (i_{n+j}(\sigma) - i_{n+j-1}(\sigma)),$$

we have

$$i_{n+j}(\sigma) = i_n(\sigma) + \frac{p^{2+m+n}e'}{p^2 - 1}(p^{2j} - 1)$$

and hence the limit  $\lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{p^{2n}}$  exists.

For depth 1 case, it means that for every  $\sigma \in G$ , there exists  $\tau \in G$  and  $n, m \in \mathbb{N}$  such that  $i_{n+j}(\sigma) < i_{m+j}(\tau) < i_{n+j+1}(\sigma)$  for all  $j \in \mathbb{N}$ . Therefore, for every  $\sigma \in G$ , we choose another  $\tau \in G$  satisfying this condition so that  $G[i_n(\sigma)] = G(u_1) = \langle \sigma^{p^n}, \tau^{p^m} \rangle$ ,  $G[i_m(\tau)] = G(u_2) = \langle \sigma^{p^{n+1}}, \tau^{p^m} \rangle$  and  $G[i_{n+1}(\sigma)] = G(u_3) = \langle \sigma^{p^{n+1}}, \tau^{p^{m+1}} \rangle$ , where  $u_1, u_2, u_3$  are three consecutive upper ramification breaks and  $G(u)^p = G(u + e)$  for all  $u \geq u_1$ . Again, let  $G' = \langle \sigma, \tau \rangle$  and let  $e'$  be the absolute ramification index of  $F'$ . We have

$$\phi_{G'}(i_{n+1}(\sigma)) - \phi_{G'}(i_n(\sigma)) = \frac{i_m(\tau) - i_n(\sigma)}{p^{n+m+1}} + \frac{i_{n+1}(\sigma) - i_m(\tau)}{p^{n+m+2}} = e'.$$

Similarly,

$$\frac{i_{n+1}(\sigma) - i_m(\tau)}{p^{n+m+2}} + \frac{i_{m+1}(\tau) - i_{n+1}(\sigma)}{p^{n+m+3}} = e'.$$

Inductively, we have

$$(3.3) \quad \frac{i_m(\tau) - i_n(\sigma)}{p^{n+m+1}} = \frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{p^{n+m+1+2j}}, \forall j \in \mathbb{N}.$$

Similarly, we can get

$$\frac{i_{n+1}(\sigma) - i_m(\tau)}{p^{n+m+2}} = \frac{i_{n+1+j}(\sigma) - i_{m+j}(\tau)}{p^{n+m+2+2j}}, \forall j \in \mathbb{N}.$$

This shows

$$\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p^2$$

for all  $n$  large enough and we also get the limit  $\lim_{n \rightarrow \infty} \frac{i_n(\sigma)}{p^{2n}}$  exists.

We summarize this result as the following.

**Theorem 3.1.** *Suppose that  $G \subseteq \text{Aut}_k(k((x)))$  is a closed subgroup which is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Then the following are equivalent:*

- (1) *For every nonidentity  $\sigma \in G$ ,  $\text{Height}(\sigma) = 2$ .*
- (2) *For every nonidentity  $\sigma \in G$ , the sequence  $\{\frac{i_n(\sigma)}{p^{2n}}\}_n$  converges.*
- (3) *For every nonidentity  $\sigma \in G$ ,  $\frac{i_{n+1}(\sigma) - i_n(\sigma)}{i_n(\sigma) - i_{n-1}(\sigma)} = p^2$  for all  $n$  sufficiently large.*
- (4) *The  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension corresponding to  $G$  is of characteristic 0.*

*Remark 3.2.* It is reasonable to extend Theorem 3.1 to the case that  $G$  is a closed subgroup of  $\text{Aut}_k(k((x)))$  which is isomorphic to a free  $\mathbb{Z}_p$ -module of rank  $n > 2$ . Our method seems not applicable to show that if every nonidentity element of  $G$  has height  $n$ , then  $G$  corresponds to an extension of characteristic 0. However, in [1], we use different approach to show that the corresponding statements (3) and (4) are equivalent.

#### 4. RAMIFICATION INDEX

Suppose that  $G$  is isomorphic to  $\mathbb{Z}_p$  with generator  $\sigma$  and is corresponding to a  $\mathbb{Z}_p$ -extension  $E/F$  of characteristic 0. Then we can use the limit  $\lim_{n \rightarrow \infty} i_n(\sigma)/p^n$  to determine the absolute ramification index  $e$  of  $F$ . In fact, by (2.1) we have  $e = \frac{p-1}{p} \lim_{n \rightarrow \infty} i_n(\sigma)/p^n$ . For the case that  $G$  is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$  with  $G = \langle \sigma, \tau \rangle$  and is corresponding to a  $\mathbb{Z}_p \times \mathbb{Z}_p$ -extension of characteristic 0, both limits  $\lim_{n \rightarrow \infty} i_n(\sigma)/p^{2n}$  and  $\lim_{n \rightarrow \infty} i_n(\tau)/p^{2n}$  exist, so it is interesting to know whether it is possible to determine the absolute ramification index by purely using the limits  $\lim_{n \rightarrow \infty} i_n(\sigma)/p^{2n}$  and  $\lim_{n \rightarrow \infty} i_n(\tau)/p^{2n}$ .

For the case of depth 2, if  $G(N)$  is generated by  $\sigma^{p^n}, \tau^{p^m}$  and  $G(u)^p = G(u + e)$  for  $u \geq N$ , then as indicated above

$$i_{n+j}(\sigma) = i_n(\sigma) + \frac{p^{2+m+n}e}{p^2 - 1}(p^{2j} - 1), \forall j \in \mathbb{N},$$

and hence

$$\lim_{j \rightarrow \infty} \frac{i_j(\sigma)}{p^{2j}} = \frac{p^2}{p^2 - 1} p^{m-n} e.$$

Similarly,

$$\lim_{j \rightarrow \infty} \frac{i_j(\tau)}{p^{2j}} = \frac{p^2}{p^2 - 1} p^{n-m} e.$$

Therefore, we have

$$e = \frac{p^2 - 1}{p^2} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\tau)}{p^{2j}}} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\sigma)}{p^{2j}}}.$$

Notice that in this case, since  $i_{n+j}(\sigma) = i_{m+j}(\tau)$  for all  $j \in \mathbb{N}$ , if we set

$$\lim_{j \rightarrow \infty} \frac{i_j(\sigma)}{p^{2j}} = \gamma_1, \lim_{j \rightarrow \infty} \frac{i_j(\tau)}{p^{2j}} = \gamma_2,$$

then  $\gamma_1/\gamma_2 = p^{2(m-n)}$ . In other words,  $\log_p \gamma_1 - \log_p \gamma_2$  must be an even number.

**Example 4.1.** For an odd prime  $p$ , let  $F = \mathbb{Q}_p(\zeta)$  be the unramified extension of degree 2 over  $\mathbb{Q}_p$  with  $\zeta$  being a unit in  $\mathcal{O}_F$ . Consider the Lubin-Tate formal group over  $\mathcal{O}_F$  constructed by  $[p](x) = px + x^{p^2}$ . For  $\alpha \in \mathcal{O}_F^*$ , let  $[\alpha](x) \in \mathcal{O}_F[[x]]$  be the automorphism of the Lubin-Tate formal group with leading coefficient  $\alpha$  and we denote its reduction by  $\sigma_\alpha \in \mathbb{F}_{p^2}[[x]]$ . For  $\alpha \in \mathcal{O}_F^*$  with  $v_F(\alpha - 1) = r$ , it is well-known that  $i_n(\sigma_\alpha) = p^{2(r+n)} - 1$  and hence we have  $\lim_{j \rightarrow \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}} = p^{2r}$ . For the case that  $\alpha = 1 + p$  and  $\beta = 1 + \zeta p$ , we know that the closed subgroup generated by  $\sigma_\alpha, \sigma_\beta$  corresponds to an extension  $N/M$  where  $M$  is the extension of  $F$  generated



by the  $p$ -torsion elements, i.e elements that satisfy  $[p](x) = 0$ . Therefore, we have the ramification index of  $M$  over  $\mathbb{Q}_p$  is  $p^2 - 1$  which is equal to

$$\frac{p^2 - 1}{p^2} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}}} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\sigma_\beta)}{p^{2j}}}.$$

Similarly, for the extension  $N/M'$  corresponding the closed subgroup  $G'$  generated by  $\alpha, \beta^p$ , we have the ramification index of  $M'$  over  $\mathbb{Q}_p$  is  $(p^2 - 1)p$ . Notice that  $G'$  is also generated by  $\sigma_\alpha, \sigma_{\beta'}$  where  $\beta' = \alpha\beta^p$ , but the ramification index of  $M'$  over  $\mathbb{Q}_p$  is not equal to

$$\frac{p^2 - 1}{p^2} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}}} \sqrt{\lim_{j \rightarrow \infty} \frac{i_j(\sigma_{\beta'})}{p^{2j}}} = p^2 - 1.$$

This is because the ramification subgroup of  $G(u)$  is not of the form  $\langle \sigma^{p^n}, \tau^{p^m} \rangle$  when  $u$  is large enough.

For the case of depth 1, if  $G(u)$  is generated by  $\sigma^{p^n}, \tau^{p^m}$  and  $G(u')^p = G(u' + e)$  for  $u' \geq u$ , then as indicated above

$$\frac{i_{m+j}(\tau) - i_{n+j}(\sigma)}{p^{n+m+1+2j}} + \frac{i_{n+1+j}(\sigma) - i_{m+j}(\tau)}{p^{n+m+2+2j}} = e, \forall j \in \mathbb{N}.$$

If we set

$$\lim_{j \rightarrow \infty} \frac{i_j(\sigma)}{p^{2j}} = \gamma_1, \lim_{j \rightarrow \infty} \frac{i_j(\tau)}{p^{2j}} = \gamma_2,$$

then

$$e = \frac{p-1}{p^{m-n+1}} \gamma_1 + \frac{p-1}{p^{n-m+2}} \gamma_2.$$

Moreover, without lose of generality we assume that  $i_{n+j}(\sigma) < i_{m+j}(\tau) < i_{n+j+1}(\sigma)$  for all  $j \in \mathbb{N}$ . Diving by  $p^{2(n+j)}$  and taking limits, we get

$$\gamma_1 \leq p^{2(m-n)} \gamma_2 \leq p^2 \gamma_1.$$

However,  $(i_{m+j}(\tau) - i_{n+j}(\sigma))/p^{n+m+1+2j}$  is a nonzero constant  $c$  for all  $j \in \mathbb{N}$  (by (3.3)). Taking limits, we get

$$p^{2(m-n)} \gamma_2 - \gamma_1 = p^{m-n+1} c \neq 0.$$

Similarly,  $p^{2(m-n)} \gamma_2 \neq p^2 \gamma_1$ . In other words,  $\log_p \gamma_1 - \log_p \gamma_2$  cannot be an even number and  $m - n$  is the unique integer between  $(1/2)(\log_p \gamma_1 - \log_p \gamma_2)$  and  $1 + (1/2)(\log_p \gamma_1 - \log_p \gamma_2)$ .

**Example 4.2.** For an odd prime  $p$ , let  $F = \mathbb{Q}_p(\pi)$  be the totally ramified extension of degree 2 over  $\mathbb{Q}_p$  with  $\pi$  being a prime element in  $\mathcal{O}_F$ . Consider the Lubin-Tate formal group over  $\mathcal{O}_F$  constructed by  $[\pi](x) = \pi x + x^p$ . For  $\alpha \in \mathcal{O}_F^*$ , let  $[\alpha](x) \in \mathcal{O}_F[[x]]$  be the automorphism of the Lubin-Tate formal group with leading coefficient  $\alpha$  and we denote its reduction by  $\sigma_\alpha \in \mathbb{F}_p[[x]]$ . For  $\alpha \in \mathcal{O}_F^*$  with  $v_F(\alpha - 1) = r$ , it is well-known that  $i_n(\sigma_\alpha) = p^{(r+2n)} - 1$  and hence we have  $\lim_{j \rightarrow \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}} = p^r$ . For the case that  $\alpha = 1 + \pi$  and  $\beta = 1 + \pi^2$ , consider  $G$  being the closed subgroup generated by  $\sigma_\alpha, \sigma_\beta$ . We have

$$\gamma_1 = \lim_{j \rightarrow \infty} \frac{i_j(\sigma_\alpha)}{p^{2j}} = p, \gamma_2 = \lim_{j \rightarrow \infty} \frac{i_j(\sigma_\beta)}{p^{2j}} = p^2.$$

Moreover,  $p-1$  is the first upper ramification break of  $G$  where  $G[p-1] = G(p-1) = G$  is generated by  $\sigma_\alpha, \sigma_\beta$  and  $G(u)^p = G(u+e)$  for  $u \geq p-1$ . Notice that  $m-n=1-1=0$  is the only integer between  $(1/2)(\log_p \gamma_1 - \log_p \gamma_2) = -1/2$  and  $1+(-1/2)$ . On the other hand,  $G$  corresponds to an extension  $N/M$  where  $M$  is the extension of  $F$  generated by the  $\pi$ -torsion elements, i.e elements that satisfy  $[\pi](x) = 0$ . Therefore, we have the ramification index of  $M$  over  $F$  is  $p-1$  and hence the ramification index of  $M$  over  $\mathbb{Q}_p$  is  $2(p-1)$  which is equal to

$$\frac{p-1}{p^{1-1+1}}p + \frac{p-1}{p^{1-1+2}}p^2.$$

Similarly, for the extension  $N/M'$  corresponding the closed subgroup  $G'$  generated by  $\alpha, \beta^p$ , we have the ramification index of  $M'$  over  $\mathbb{Q}_p$  is  $2(p-1)p$ .

We summarize our result as the following.

**Theorem 4.3.** *Let  $G$  be a closed subgroup of  $\text{Aut}_k(k((x)))$  which is isomorphic to  $\mathbb{Z}_p \times \mathbb{Z}_p$ . Suppose  $G$  corresponds to an extension of characteristic 0. Suppose further that  $G = \langle \sigma, \tau \rangle$ ,  $G(u) = \langle \sigma^{p^n}, \tau^{p^m} \rangle$  and  $G(u')^p = G(u'+e)$  for all  $u' \geq u$ . Let*

$$\gamma_1 = \lim_{j \rightarrow \infty} \frac{i_j(\sigma)}{p^{2j}}, \gamma_2 = \lim_{j \rightarrow \infty} \frac{i_j(\tau)}{p^{2j}}.$$

(1) *If  $\log_p(\gamma_1/\gamma_2)$  is an even number, then  $G$  is of depth 2 and*

$$e = \frac{p^2-1}{p^2} \sqrt{\gamma_1 \gamma_2}.$$

(2) *If  $\log_p(\gamma_1/\gamma_2)$  is not an even number, then  $G$  is of depth 1. Furthermore, let  $a$  be the unique integer between  $(1/2) \log_p(\gamma_1/\gamma_2)$  and  $1 + (1/2) \log_p(\gamma_1/\gamma_2)$ . Then*

$$e = \frac{p-1}{p^{a+1}} \gamma_1 + \frac{p-1}{p^{2-a}} \gamma_2.$$

**Remark 4.4.** In the case of depth 2, let  $a$  be the integer  $(1/2) \log_p(\gamma_1/\gamma_2)$ . Then we have

$$\frac{p^2-1}{p^2} \sqrt{\gamma_1 \gamma_2} = \frac{p-1}{p^{a+1}} \gamma_1 + \frac{p-1}{p^{2-a}} \gamma_2.$$

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